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Anisotropic cosmological models with spinor field and viscous fluid in the presence of a Λ term: qualitative solutions

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Abstract

The study of a self-consistent system of nonlinear spinor and Bianchi type I gravitational fields in the presence of a viscous fluid and a Λ term, with the spinor field nonlinearity being some arbitrary functions of the invariants I and J constructed from bilinear spinor forms S and P, generates a multi-parametric system of ordinary differential equations (Saha 2005 *Rom. Rep. Phys.* **57** 7, Saha 2007 *Preprint* gr-qc/0703085 (*Astrophys. Space Sci.* at press)). A qualitative analysis of the system in question has been thoroughly carried out. A complete qualitative classification of the mode of the evolution of the universe given by the corresponding dynamic system has been illustrated.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Though the investigation of relativistic cosmological models usually has the energy–momentum tensor of matter generated by a perfect fluid, to consider more realistic models one must take into account the viscosity mechanisms [3–7]. On the other hand, in the recent years anisotropic cosmological models with nonlinear spinor field have been extensively studied due to the facts that (i) the introduction of the nonlinear spinor field into the system leads to the isotropization of initially anisotropic universe [8–10]; (ii) the spinor field nonlinearity in some cases can give rise to singularity-free solutions [10, 11]; (iii) the spinor field can be considered as one of the possible candidates to explain the late time acceleration of the universe [12, 13].

Given the importance of the viscous fluid and spinor field to model a realistic universe, recently we have considered a self-consistent system of the nonlinear spinor field and a gravitational field described by a Bianchi type I (BI) cosmological model filled with the viscous fluid [1, 2]. In [1, 2], we have thoroughly studied the corresponding field equations. Exact solutions of the field equations were given in terms of the volume scale of the BI spacetime τ. Multi-parametric system of equations for the volume scale τ, the energy density of the viscous fluid ε were solved for some special choice of bulk and shear viscosities. Given the richness of the system mentioned above in this paper we study it qualitatively for more general cases. In the absence of the viscosity the system allows integrals of motion, whereas it is either impossible to obtain, or there is no first integral at all in general when the viscosity is taken into account. As a result, the study of the possible modes of the evolution becomes very difficult. Undoubtedly, the result of the investigation should be presented in the form of numerical values and at the same time cannot be reduced to a representation of some causally found examples of numerical solutions. Clearly, it should be a classification of modes of the evolution in the parametric space. Actually, it is the task of the qualitative analysis.

2. Basic equations

We consider a self-consistent system of nonlinear spinor and BI gravitational fields filled with a viscous fluid in the presence of a cosmological term. As it was shown in [1, 2], the components of the spinor field and metric functions can be expressed in terms of the volume scale τ of the BI spacetime. So one needs to find the function τ explicitly. The corresponding equation can be derived from Einstein equations and Bianchi identity (a detailed description of this procedure can be found in [1, 2]). For convenience, we also define the generalized Hubble constant. The system then reads

$$\dot{\tau} = 3H\tau, \tag{2.1a}$$

$$\dot{H} = \frac{1}{2}(3\xi H - \omega) - (3H^2 - \kappa\varepsilon - \Lambda) + \frac{\kappa}{2}\left(\frac{m}{\tau} + \frac{\lambda(n-2)}{\tau^n}\right), \tag{2.1b}$$

$$\dot{\varepsilon} = 3H(3\xi H - \omega) + 4\eta(3H^2 - \kappa\varepsilon - \Lambda) - 4\eta\kappa\left[\frac{m}{\tau} - \frac{\lambda}{\tau^n}\right], \tag{2.1c}$$

where

$$\omega = \varepsilon + p \tag{2.2}$$

is the thermal function. Here κ is the Einstein's gravitational constant, Λ is the cosmological constant, λ is the self-coupling constant, m is the spinor mass and n is the power of nonlinearity of the spinor field (here we consider only power-law nonlinearity). In (2.1), η and ξ are the bulk and shear viscosities, respectively, and they are both positively definite, i.e.,

$$\eta > 0, \qquad \xi > 0. \tag{2.3}$$

They may be either constant or function of time or energy. We consider the case when

$$\eta = A\varepsilon^{\alpha}, \qquad \xi = B\varepsilon^{\beta}, \tag{2.4}$$

with A and B being some positive quantities. For p we set as in perfect fluid,

$$p = \zeta\varepsilon, \qquad \zeta \in (0, 1]. \tag{2.5}$$

Note that in this case $\zeta \neq 0$, since for dust pressure, hence, temperature is zero, that results in vanishing viscosity. Note that a system in the absence of the spinor field has been studied in [14, 15]. In that case, the corresponding system is analogical to that given in (2.1) without the third terms in (2.1b) and (2.1c).

3. Qualitative analysis

Research on the behavior of the dynamic system given by a system of ordinary differential equations implies the survey of all possible scenarios of development for different values of the problem parameters. It is necessary to understand at least how the process of the evolution comes to an end if it does so at infinitively large time for a given set of initial conditions which can be given anywhere.

So, under the specific behavior of the system we understand the phase portrait of the system, i.e., the family of integral curves, covering the total phase space. It is easy to imagine as far as any point of the space can be declared as the initial one and at least one integral curve will pass through it (or it will be fixed point).

Certainly, it is difficult to imagine such a set of curves. In many cases, close (and not only) curves transform into each other at some diffeomorphism of space. These curves are known as topologically equivalent. The differences between them are not very important for our study. They all behave in the same manner. This relation—'the relation of equivalence'—divides the family of curves into the classes of equivalence. For graphical demonstration it will be convenient to present at least one representative of each class.

The change of the value of problem parameters not always results in a significant change of the phase portrait. Repeating this method, we say that one family of integral curves (covering the total space) for the given set of parameters is equivalent to the other for another set of parameters, if there exists a diffeomorphism of space transforming the first family into the second. It is clear that there occurs the division into the classes of equivalence, and we are not very interested in differences between equivalent families. We argue that the corresponding changes in parameters do not alter anything on principle. So it is sufficient to demonstrate only one phase portrait for a given set of parameters underlining the features of the given class.

However, for some critical relations between the parameters there occur significant changes. These are the boundary relations of parameters, dividing, as usual, the parameter space into regions of similar behavior. Thus accomplishing the qualitative classification of the mode of the evolution of the dynamic system. Now, giving the concrete value of parameters, we can define which region of parameters they correspond to, thus defining the type of behavior. Moreover, given the specific initial conditions, we can answer the question to which region of the phase space the evolution of the system leads in time.

In our cosmological model, numerical parameters A, α, B, β are related to the viscosity, while λ and Λ are the (self)-coupling and cosmological constants.

Initially, we consider the system of Einstein and Dirac equations. Solving these equations, we find the components of the spinor field and metric functions a, b, c in terms of the volume scale $\tau = abc$ of the BI universe. Finally, in order to find τ from Einstein equations and Bianchi identity, we deduce three first-order ordinary differential equations. Further for convenience we introduce a new function ν inverse to τ, i.e., $\nu = 1/\tau$.

The fact that the system has the dimension greater than 2 strongly complicates qualitative analysis. Note that the well-known Lorenz system of three ordinary differential equations with polynomial on the right-hand side with degree less or equal to 2 possesses in some region of the parameter space chaotic behavior known as a strange attractor and in that region there do not exist first integrals (i.e., globally defined invariants). Though the set of singularities is very simple, there exist only three singular (fixed) points: two focus and one saddle. The presence of such an example does not allow us to make an optimistic conclusion on the basis of simple construction of our system (with polynomials on the right-hand side and in the absence of singular points in the region of space we are interested in, which is even dynamically closed).

Nevertheless, on the boundary of the space $\epsilon = 0$, as well as $\nu = 0$ ($\tau = +\infty$), which are dynamically closed themselves, the complete classification has been done. The dynamical closeness of these planes is an obstacle for penetration from positive octant $\epsilon > 0 \wedge \nu > 0$ to the region with negative values. But, there are no singularities, fixed points (there are fixed points on the boundary) in the positive octant, we were not able to prove the simplicity of its behavior, e.g., the presence of first integrals, as well as their absence.

Thus let us go back to the system (2.1) in detail. As it was already mentioned, it is convenient to define a new function $\nu = 1/\tau$. In this case, the obvious singularity that occurs at $\tau = 0$ vanishes and $\nu = 0$ corresponds to $\tau = \infty$ while $\nu = \infty$ to $\tau = 0$. The system (2.1) on account of (2.4) takes the form

$$\dot{\nu} = -3H\nu, \tag{3.1a}$$

$$\dot{H} = \tfrac{1}{2}(3B\varepsilon^{\beta}H - (1+\zeta)\varepsilon) - (3H^2 - \varepsilon - \Lambda) + \tfrac{1}{2}(m\nu + \lambda(n-2)\nu^{n-1}), \tag{3.1b}$$

$$\dot{\varepsilon} = 3H(3B\varepsilon^{\beta}H - (1+\zeta)\varepsilon) + 4A\varepsilon^{\alpha}(3H^2 - \varepsilon - \Lambda) - 4A\varepsilon^{\alpha}[m\nu - \lambda\nu^{n}]. \tag{3.1c}$$

Let us now study the foregoing system of equations in detail.

3.1. Behavior of the solutions on the $\nu = 0$ plane

Let us first study the behavior of the functions H and τ on the $\nu = 0$ plane. The plane $\nu = 0$ is dynamically invariant, since $\dot{\nu}|_{\nu=0} = 0$. It should be emphasized that the system in this case coincides with that in the absence of the spinor field and was thoroughly studied in [15]. Nevertheless, we write the results obtained in detail. In doing so we rewrite equations (3.1*b*) and (3.1*c*) in the matrix form

$$\begin{pmatrix} \dot{H} \\ \dot{\varepsilon} \end{pmatrix} = \begin{pmatrix} \kappa/2 & -1 \\ 3H & 4\eta \end{pmatrix} \begin{pmatrix} 3B\varepsilon^{\beta}H - (1+\zeta)\varepsilon \\ 3H^2 - \kappa\varepsilon - \Lambda \end{pmatrix}. \tag{3.2}$$

(a) By virtue of linear independence of the columns of the matrix of equation (3.2) the critical points are the solutions of the equations

$$3B\varepsilon^{\beta}H - (1+\zeta)\varepsilon = 0, \tag{3.3a}$$

$$3H^2 - \kappa\varepsilon - \Lambda = 0, \tag{3.3b}$$

i.e., they necessarily lie on the parabola (3.3*b*). In view of

$$H = \frac{1+\zeta}{3B}\varepsilon^{1-\beta},$$

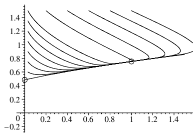
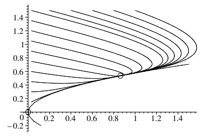
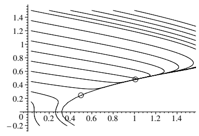
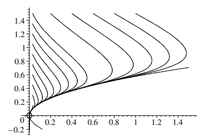
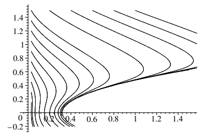
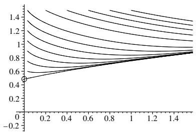
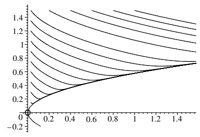
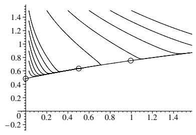
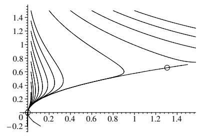
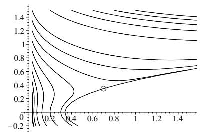
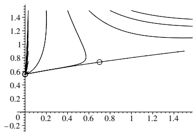
which follows from (3.3*a*), equation (3.3*b*) can be written as

$$3\kappa B^2\varepsilon^{1+2\beta} - (1+\zeta)^2\varepsilon^2 + 3\Lambda B^2\varepsilon^{2\beta} = 0. \tag{3.4}$$

The solutions to the system (3.3) will be the roots of equation (3.4). The quantity of the positive roots of equation (3.4) according to the Cartesian law is equal to the number of changes of sign of the coefficients of equations or less than that by an even number. So, for $\Lambda > 0$ and $1/2 < \beta < 1$ or $\Lambda < 0$ and $\beta < 1/2$ the number of roots is either 2 or 0. For the remaining cases, i.e., $\Lambda > 0$ and $\beta > 1$ or $\Lambda > 0$ and $\beta < 1/2$ or $\Lambda < 0$ and $\beta > 1/2$, there exists only one root.

In table 1, the classification of qualitatively different types of evolution (phase portrait) depending on the parameters β, Λ and $(1+\zeta)/B$ is illustrated. Figure (a) in table 1 corresponds to the two types, namely $\beta < 1/2$ or $\beta = 1/2$ and $(1+\zeta)/B < \sqrt{3\kappa}$, as well as figure (i)

Table 1. Classification of qualitatively different types of evolution (phase portrait) depending on the parameters β, Λ and $(1+\zeta)/B$.

		$\Lambda > 0$	$\Lambda = 0$	$\Lambda < 0$
$\beta < 1/2$		 (*a*)	 (*b*)	
$\beta = 1/2$	$\frac{1+\zeta}{B} > \sqrt{3\kappa}$		 (*c*)	 (*d*)
	$\frac{1+\zeta}{B} < \sqrt{3\kappa}$		 (*f*)	
$1/2 < \beta < 1$		 (*e*)	 (*g*)	 (*h*)
$\beta = 1$	$\frac{1+\zeta}{B} > 3\Lambda$			
	$\frac{1+\zeta}{B} < 3\Lambda$	 (*i*)		
$\beta > 1$				

to the cases $\beta > 1$ or $\beta = 1$ and $(1+\zeta)/B < \sqrt{3\kappa}$. Figures (g) and (h) of table 1 cover all the four cases for $\beta > 1/2$. Figures (d) and (e) contain pairs of graphics (case with 0 or 2 singular points; case with 1 singular point is also allowable since it possesses the frequency 2, i.e., two singular points merges to one saddle-knot and no other qualitative change takes place). They cover three cases: (d) corresponds to $\beta \leqslant 1/2$; (e) corresponds to $\beta = 1/2$ and $(1+\zeta)/B < \sqrt{3\kappa}$ or $1/2 \leqslant \beta \leqslant 1$ and $\beta = 1/2$ and $(1+\zeta)/B > \sqrt{3\kappa}$.

Since, the equation for ε only contains η, the energy density for nontrivial η undergoes essential changes, whereas H and τ remain virtually unchanged.

As far as the critical points saddle and attracting knots lie on the integral curve alternately, so it is sufficient to consider the case with maximum number of roots. Taking into account equations (2.1c) and (3.3b) we calculate

$$\lim_{\varepsilon\to+\infty}\frac{\dot\varepsilon}{3H\varepsilon} = \lim_{\varepsilon\to+\infty}\frac{\sqrt{3}B\varepsilon^{\beta}\sqrt{\kappa\varepsilon+\Lambda}-\varepsilon(1+\zeta)}{\varepsilon}$$

$$= \sqrt{3}B\sqrt{\kappa\varepsilon^{(2\beta-1)}+\Lambda\varepsilon^{-2}}-(1+\zeta) = \begin{cases} -(1+\zeta)<0, & \beta<1/2,\\ B\sqrt{3\kappa}-(1+\zeta), & \beta=1/2,\\ +\infty>0, & \beta>1/2.\end{cases} \quad (3.5)$$

So, the latest critical point for $\beta < 1/2$ is attracting knot while for $\beta > 1/2$ it is a saddle. In the case of $\beta = 1/2$, we have either a saddle or an attracting knot depending on the sign (positivity to negativity) of $B\sqrt{3\kappa}-(1+\zeta)$.

(b) In the case of $\Lambda \geqslant 0$ the points of intersection of the boundary are the critical points

$$H = \pm\sqrt{\Lambda/3}, \quad (3.6a)$$

$$\varepsilon = 0. \quad (3.6b)$$

(c) For $H < 0$ there may exist critical points, if the columns of the matrix of (3.2) are linearly dependent. In that case, the critical points are the roots of the equation

$$3\kappa(\zeta-1)\varepsilon + 6\kappa^2 AB\varepsilon^{\alpha+\beta} + 8\kappa^2A^2\varepsilon^{2\alpha} - 6\Lambda = 0 \quad (3.7)$$

and

$$H = -\tfrac{2}{3}\kappa A\varepsilon^{\alpha}. \quad (3.8)$$

In the case of $\eta = 0$, the roots of the characteristic equation

$$\left|\frac{D(\dot H,\dot\varepsilon)}{D(H,\varepsilon)} - \mu\right| = 0 \quad (3.9)$$

are

$$\mu_{1,2} = \frac{3\kappa\xi \pm \sqrt{9\kappa^2\xi^2 + 48\Lambda(1+\zeta)}}{4}. \quad (3.10)$$

The critical point $(H,\varepsilon) = (0, 2\Lambda/[\kappa(\zeta-1)])$ is of type divergent focus if $\Lambda > -9\kappa^2\xi^2/[48(1+\zeta)]$ or divergent knot if $\Lambda < -9\kappa^2\xi^2/[48(1+\zeta)]$.

3.1.1. Integral curves. For $\Lambda \geqslant 0$ the solutions starting from the upper half-plane $H > 0$ cannot enter into the lower one. For $\Lambda < 0$ some of the solutions may enter into the lower half-plane through the segment $H = 0$ and $\Lambda \leqslant 0 \leqslant \varepsilon$ and never returns back, since $\dot H|_{H=0} < 0$.

3.2. Behavior of the solutions on the $\varepsilon = 0$ plane

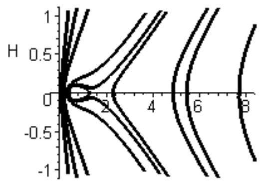
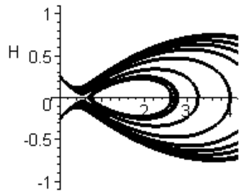
The plane $\varepsilon = 0$ is dynamicly invariant, since $\dot\varepsilon|_{\varepsilon=0} = 0$. Depending on the sign of H this plane is either attractive or repulsive, namely for $H > 0$ it is attractive and for $H < 0$ it is repulsive, since

$$\frac{\partial\dot\varepsilon}{\partial\varepsilon} = -3H(1+\zeta) < 0.$$

On the $\varepsilon = 0$ plane the system (3.1) takes the form

$$\dot\nu = -3H\nu, \quad (3.11a)$$

Table 2. Classification of qualitatively different types of evolution (phase portrait) on the $\varepsilon = 0$ plane for $n = 3$.

$\Lambda < 0$	$\Lambda > 0$
	
(*a*)	(*b*)

$$\dot{H} = -3H^2 + \Lambda + \tfrac{1}{2}(m\nu + \lambda(n-2)\nu^{n-1}). \tag{3.11b}$$

The system (3.11) has the following integrals:

$$3H^2 = C\nu^2 + m\nu + \Lambda - \frac{\lambda(n-2)\nu^{n-1}}{n-3}, \qquad n > 3, \tag{3.12a}$$

$$3H^2 = C\nu^2 + m\nu + \Lambda - \nu^2 \ln(\nu), \qquad n = 3, \tag{3.12b}$$

$$3H^2 = C\nu^2 + m\nu + \Lambda, \qquad n = 2, \tag{3.12c}$$

where C is some arbitrary constant.

The characteristic equation of nontrivial singular points on the $\varepsilon = 0$ plane for the system (2.1) takes the form

$$\lambda(n-2)\nu^{n-1} + m\nu + 2\Lambda = 0. \tag{3.13}$$

Depending on changes of signs in the sequence of λ, m, Λ it has one, two or no solutions.

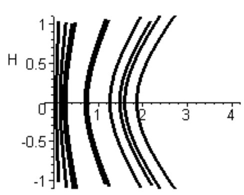
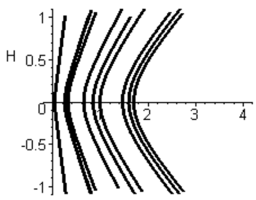
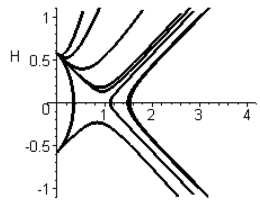
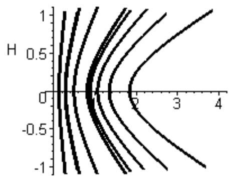
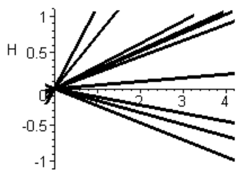
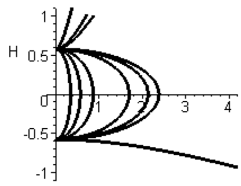
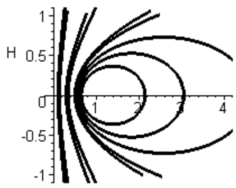
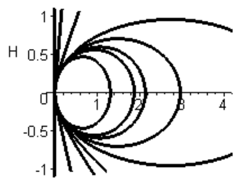
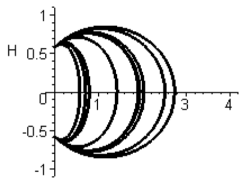
In table 2 we illustrated the phase portrait on the $\varepsilon = 0$ plane for a positive and a negative Λ, respectively, for $n = 3$.

In table 3, we have graphically illustrated the H–ν phase portrait on the $\varepsilon = 0$ plane for different Λ. As it was mentioned earlier, here we deal with the multi-parametric system of ordinary nonlinear differential equation. In doing so we consider all possible variants independent of their physical validity. Therefore, we demonstrate the results obtained for a negative spinor mass ($m < 0$).

If the right-hand side of (3.12) possesses two positive roots with H being positive between them, then on the plane $\varepsilon = 0$ there occurs closed cycle. It is obvious that there can be no more than three roots, hence there cannot be non-concentric cycles. As a result, near the plane $\varepsilon = 0$ there might be cyclic oscillations.

The singular point around which the oscillation takes place has $H = 0$ and, therefore, the trajectory of oscillation partially passes in the region which is attractive to the plane $\varepsilon = 0$ and partially in the region that is repulsive. In the long run in the repulsive region at some moment the growth of ε becomes dominant. It results in the fact that ε becomes infinity within a finite range of time.

Table 3. Classification of qualitatively different types of evolution (phase portrait) on the $\varepsilon = 0$ plane for $n = 2$.

	$\Lambda < 0$	$\Lambda = 0$	$\Lambda > 0$
$m < 0$	 (*a*)	 (*b*)	 (*c*)
$m = 0$	 (*d*)	 (*e*)	 (*f*)
$m > 0$	 (*g*)	 (*h*)	 (*i*)

3.2.1. Invariants of evolution. The system (3.1) in the absence of the viscosity, i.e., under $\eta = 0$ and $\xi = 0$, possesses the following first integrals:

$$F_1 = \frac{\varepsilon}{\nu^{1+\zeta}}, \tag{3.14a}$$

$$F_2 = \frac{(H^2 - \varepsilon - \Lambda - m\nu)}{\nu^2} + \lambda \frac{n-2}{n-3}\nu^{n-3}. \tag{3.14b}$$

The first of them (3.14*a*) remains to be the first integral even after the introduction of the bulk viscosity ξ. The second one, i.e., equation (3.14*b*) under $\xi \neq 0$, ceases to be the integral of motion. Nevertheless, the introduction of the bulk viscosity during the course of time generates definite displacement of the surface given by formula (3.14*b*), which allows one

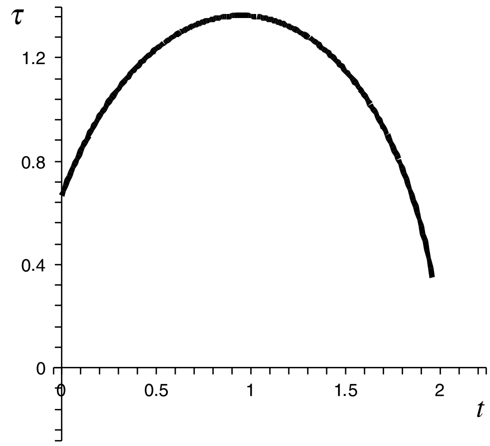


Figure 1. Evolution of the volume scale $\tau = 1/\nu$.

qualitatively, i.e., based only on the continuity, compile the representation about the possible ways of evolution.

3.3. Qualitative analysis of the complete system

Harnessing tables 2, 3 as well as 1 helps one to understand the 3D phase portrait leaning on the continuous dependence of the velocity fields of the coordinates ν, H, ε of the phase space.

In order to cover the infinite phase space completely, it is mapped on coordinate parallelepiped with its axes being the arc-tangent of the corresponding coordinates. The lower horizontal plane always represents the $\varepsilon = 0$ plane.

It should be noted that the introduction of the spinor field notably complicates the evolution of the system. Contrary to the system in the absence of the spinor field, the initial condition with $H < 0$ already does not prevent in many cases thanks to the evolution of the volume scale entering the half-space $H > 0$ and thereupon, from the greater value of H repeats the evolution, approaching to the $\nu = 0$ plane and displaying the classification from table 1. In the vicinity of the borders $\varepsilon = 0$ and $\nu = 0$ the integral curves closely repeat the integral curves on the sides, each time at least to some extent.

The general property of all the cases is the fact that in the half-space $H > 0$ the velocity vectors are directed to the $\varepsilon = 0$ plane, while in the other half opposite to it. As a result all the invariant curves fall on $\varepsilon = 0$, though not necessarily reach it.

In figures 1–18, we have illustrated the volume scale $\tau(t)$ (figures 1, 4, 7, 10, 13 and 16), the energy density $\varepsilon(t)$ (figures 2, 5, 8, 11, 14 and 17) and the phase portrait in the ν, H, ε space (figures 3, 6, 9, 12, 15 and 18). Figures 1–12 correspond to the positions c, g, h and i of table 3. In doing so we used the following values of the parameters: $\alpha = 4, \beta = 1, \zeta = 0.5, A = 1, B = 1$ and $n = 4$. The positions h and i correspond to the geometrically cyclic regime, but the case h possesses fixed point on the cyclic integral curve, hence corresponds to the intermediate stage between periodic and non-periodic. The positions c and g correspond to the non-periodic evolution.

The bold black line in the 3D figures (figures 3, 6, 9, 12, 15 and 18) corresponds to the functions $\tau(t)$ and $\varepsilon(t)$ presented in the preceding figures (figures 1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17). Figures 1–3 and 10–12 correspond to the case k of table 1, figure 4–6 to i and figure 7–9 to h, respectively.

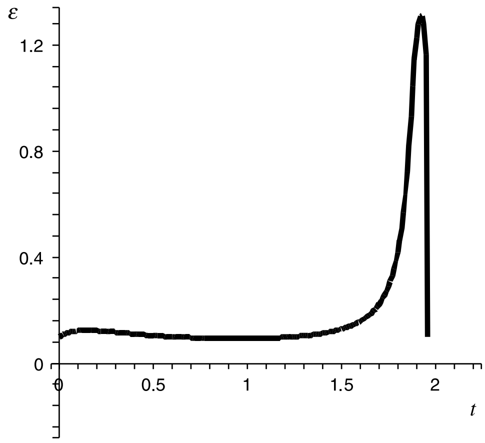


Figure 2. Evolution of the energy density.

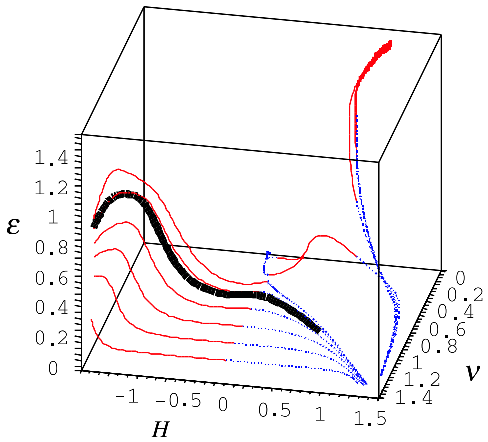


Figure 3. 3D view in the ν, H, ε space.

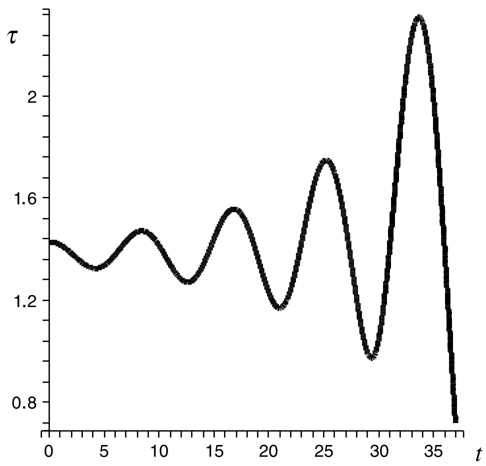


Figure 4. Evolution of the volume scale $\tau = 1/\nu$.

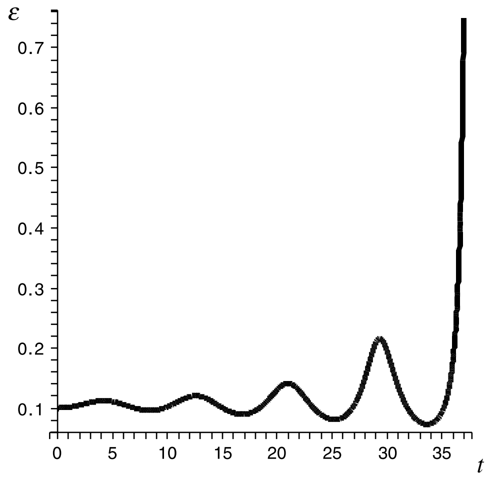


Figure 5. Evolution of the energy density.

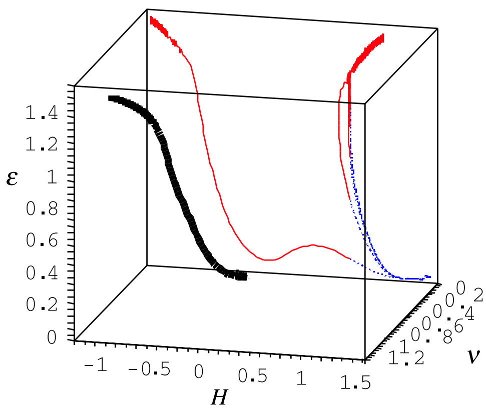


Figure 6. 3D view in the ν, H, ε space.

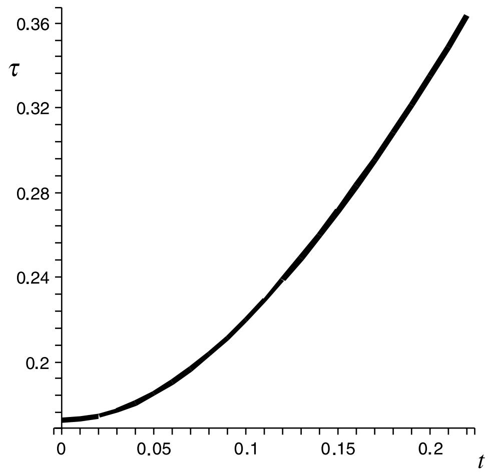


Figure 7. Evolution of the volume scale $\tau = 1/\nu$.

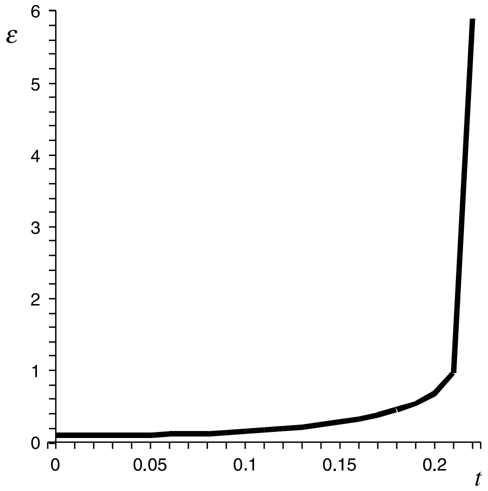


Figure 8. Evolution of the energy density.

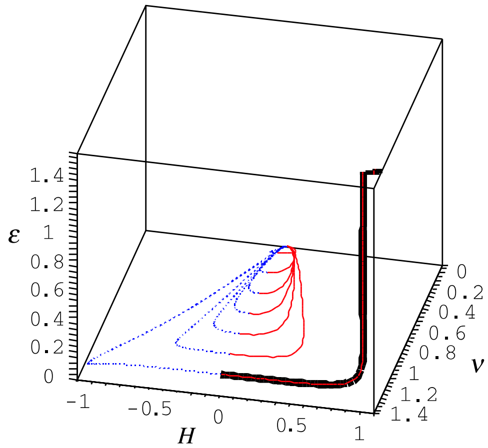


Figure 9. 3D view in the ν, H, ε space.

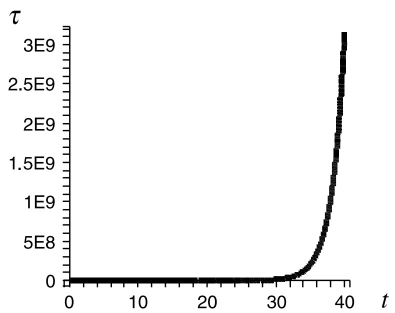


Figure 10. Evolution of the volume scale $\tau = 1/\nu$.

In figures 1–3, we have plotted τ, ε and phase portrait in the ε, H, ν space for a negative m ($m = -0.1$), a positive Λ ($\Lambda = 0.1$) and a negative self-coupling constant λ ($\lambda = -1$). Both

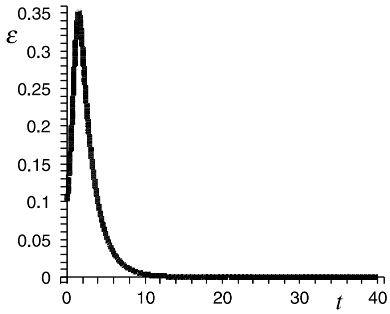


Figure 11. Evolution of the energy density.

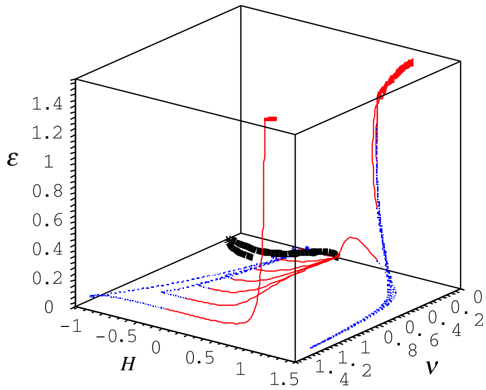


Figure 12. 3D view in the ν, H, ε space.

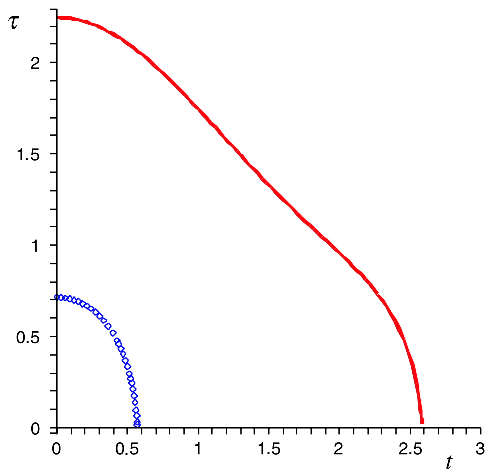


Figure 13. Evolution of the volume scale $\tau = 1/\nu$.

τ and ε initially increase and after reaching some maximum begin to decrease and ultimately halt at some finite value with H tending to $-\infty$.

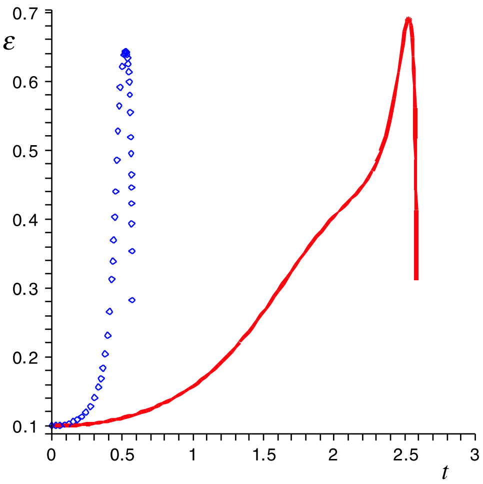


Figure 14. Evolution of the energy density.

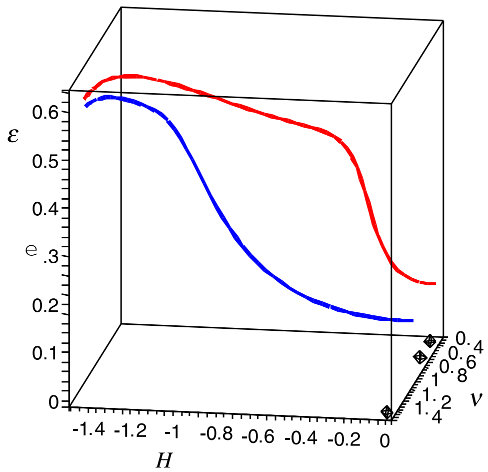


Figure 15. 3D view in the ν, H, ε space.

Figures 4–6 correspond to the case with $m = 0.1$, $\Lambda = -0.1$ and $\lambda = 0.1$. In this case, we have the oscillatory mode of expansion at the initial stage (as it may be expected for a negative Λ), but ultimately ending with a big crunch.

Figures 7–9 correspond to the parameters $m = 0.1$, $\Lambda = 0$ and $\lambda = 0.1$, i.e., case without cosmological constant. As one sees, in this case both τ and ε increase with time. H in this case is positive, increases rapidly at the initial stage, but after sometime tends to some finite value. In this case, we have future singularity similar to a big rip. It should be noted that in the case of a perfect fluid the big rip occurs with phantom dark energy whereas we come to this stage thanks to the viscous fluid and spinor field.

In figures 10–12, we have illustrated the corresponding functions and their phase portrait for $m = 0.1$, $\Lambda = 0.1$ and $\lambda = 0.1$. As was expected, the positive Λ leads to an expanding mode of evolution with the energy density tending to zero.

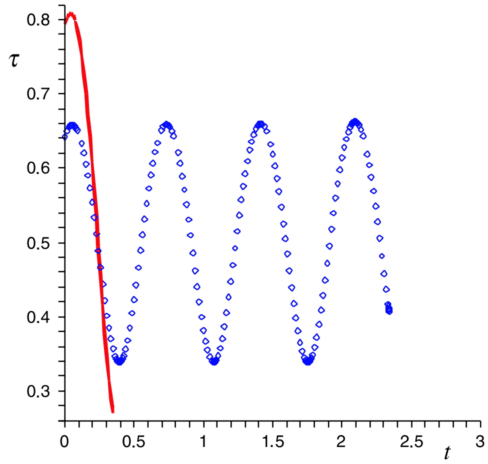


Figure 16. Evolution of the volume scale $\tau = 1/\nu$.

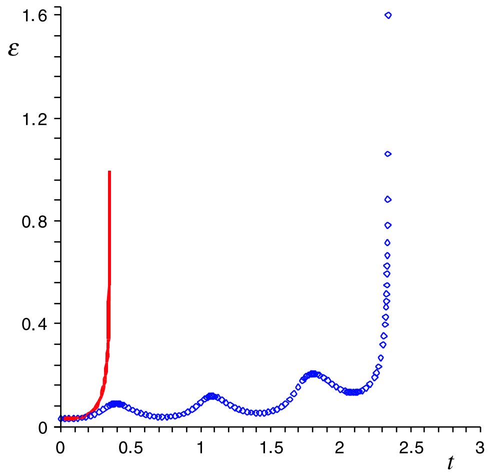


Figure 17. Evolution of the energy density.

For $m = 4$, $\Lambda = -1$ and $\lambda = -1$ we have a non-periodic mode of evolution (cf figure 13). But unlike the big crunch when at the point of spacetime singularity ($\tau = 0$) the energy density tends to ∞, in the case considered we have a maximum but finite value of ε (cf figure 14). The corresponding phase portrait is illustrated in figure 15. As one sees, independent of initial condition the universe shrinks into a point ($\tau \to 0$) in the course of evolution.

Finally, in figures 16–18 we have plotted the volume scale, energy density and phase portrait in the ε, H, ν space for $m = 1$, $\Lambda = -10$ and $\lambda = 1$. Depending on the initial conditions in this case, we have either the non-periodic or the oscillatory mode of evolution. The case of non-periodic evolution corresponds to a big crunch as was expected. For the oscillatory mode of evolution with τ being finite and nontrivial and H being finite, ε tends to ∞ within finite time.

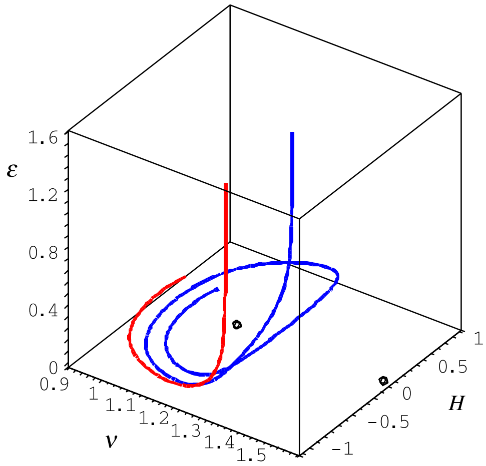


Figure 18. 3D view in the ν, H, ε space.

4. Conclusion

Recently, a self-consistent system of nonlinear spinor and gravitational fields in the framework of Bianchi type I cosmological model filled with the viscous fluid was considered by one of the authors [1, 2]. The spinor field nonlinearity is taken to be some power law of the invariants of bilinear spinor forms, namely $I = S^2 = (\bar{\psi}\psi)^2$ and $J = P^2 = (\mathrm{i}\bar{\psi}\gamma^5\psi)^2$. Solutions to the corresponding equations are given in terms of the volume scale of the BI spacetime, i.e., in terms of $\tau = abc$, with a, b, c being the metric functions. This study generates a multi-parametric system of ordinary differential equations [1, 2]. Given the richness of the system of equations in this paper a qualitative analysis of the system in question has been thoroughly carried out. A complete qualitative classification of the mode of evolution of the universe given by the corresponding dynamic system has been illustrated. In doing so, we have considered all possible values of the problem parameters and graphically presented the most distinguishable in our view results. Here we would like to stop on a few points. First of all the choice of a negative Λ term. It is well known that after the discovery of acceleration in the evolution of the universe in 1998, cosmologists all over the world rushed to explain the new phenomena. One of the simplest ways was to introduce a positive Λ term into the system. Further, many other models such as quintessence, k-essence, chaplygin gas, phantom, etc have been proposed. Unfortunately, the discovery of the positive acceleration of the universe posed a number of problems. Among them one of the most puzzling is the eternal acceleration. It was shown in [17] that a positive Λ indeed results in an eternal acceleration. Several possible solutions of this problem are based on cycles in the evolution of the universe [16] or the introduction of a negative potentials for the scalar field [18]. It can be obtained as well with a negative cosmological constant [19]. In a number of papers by one of the authors [10, 11] it was shown that the introduction of a negative Λ, which is equivalent to an additional gravitational force, into the system gives rise to an oscillatory or non-periodic mode of expansion.

Here during the qualitative analysis we have used a negative mass. Indeed, it does not really mean the spinor field possesses a negative mass. By introducing a negative mass we just put a negative sign before the mass term in the Lagrangian [1]. It should be noted that unlike the scalar field Lagrangian, where the sign before the mass term in the Lagrangian is crucial one, in the case of spinor field things are totally different. In order to obtain the Dirac equation we must require that the wavefunction ψ obeys the Klein–Gordon equation

$$(\Box - m^2)\psi = 0, \tag{4.1}$$

if it is to describe a free particle of mass m since this equation implies that the energy–momentum relation for a free particle $p^2 = m^2c^2$ is satisfied. In order that the function ψ obeys equation (4.1), we can as well demand that it also satisfies one of the following equations [20]:

$$(\mathrm{i}\gamma^\mu\partial_\mu + m)\psi = 0 \qquad \text{or} \qquad (\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = 0. \tag{4.2}$$

Both the equations can be obtained from the Lagrangian

$$L = \frac{\mathrm{i}}{2}[\bar\psi\gamma^\mu\partial_\mu\psi - \partial_\mu\bar\psi\gamma^\mu\psi] \pm m\bar\psi\psi. \tag{4.3}$$

Thus we see that the sign before the mass term in the Lagrangian (4.3) gives just other set of Dirac equation, in particular, equations for ψ and $\bar\psi$ interchanges. Here we would like to note that in the case of curve spacetime the partial derivative should be replaced by a covariant one. So from physical point of view the case with $m < 0$ in this paper is not really an exotic one.

Acknowledgments

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